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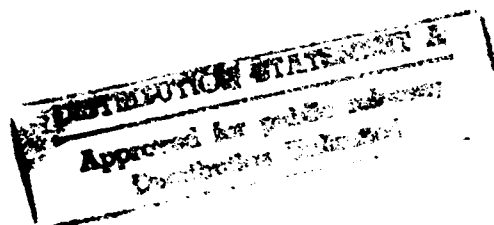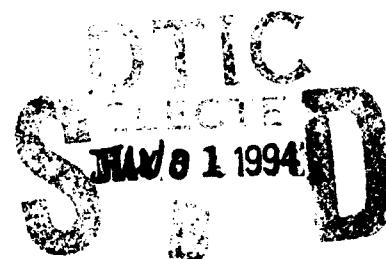
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AN ITERATIVE TECHNIQUE TO STABILIZE A LINEAR TIME INVARIANT

MULTIVARIABLE SYSTEM WITH OUTPUT FEEDBACK

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16. Abstract

The problem of finding a physically realizable, i.e. constant gain, output-feedback controller which will stabilize an unstable plant is one of the oldest and most fundamental problems in control theory. In spite of its long history, this problem still remains unsolved even in the seemingly simple case of linear plants.

In this paper, an iterative procedure for determining the constant gain matrix that will stabilize a linear constant multivariable system using output feedback is described. The use of this procedure avoids the transformation of variables which is required in other procedures. For the case in which the product of the output and input vector dimensions is greater than the number of states of the plant, we are able to give a rather general solution. In the case in which the states exceed the product of input and output vector dimensions we are able to present a least square solution which may not be stable in all cases. The results are illustrated with examples.

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1. INTRODUCTION

The design of linear multivariable control systems using output feedback has attracted the attention of many authors [Davison 1970, Fallside and Seraji 1971, Sridhar and Lindorff 1973, Fortmann 1973, Seraji 1973, and others]. There are two ways of approaching this problem. The first method consists of estimating the states of the system using an observer and using these states in the subsequent design. In the second approach, either static or dynamic feedback of the output is used directly in the control problem and this view is adopted here.

Most of the techniques discussed in the literature for approach two require the system matrix to be transformed to a canonical form which may cause some loss of physical insight. This paper describes an iterative procedure for determining the constant gain matrix that will stabilize a linear constant multivariable system using output feedback gains without the necessity of transforming variables while achieving a satisfactory degree of stability and damping ratio. The resultant control problem is algebraic and hence easy to solve.

2. PROBLEM STATEMENT

Consider a linear time-invariant controllable and observable system

$$\dot{x}(t) = A\,x(t) + Bu(t)$$
$$y(t) = C\,x(t) \tag{1}$$

where the state x, input u and output y have dimensions n, m and p, respectively. The output feedback control law is given by

$$u(t) = G\,y(t) \tag{2}$$

where G is a constant mxp feedback gain matrix. Direct substitution of (2) in (1) yields

$$\dot{x}(t) = (A + BGC) x(t)$$
$$y(t) = C x(t)$$

(3)

The stability of (3) depends primarily on the eigenvalues of A + BGC. If all the eigenvalues have real parts strictly less than $-\nu$ for $\nu > 0$, then the system will be said to have degree of stability ν. The damping ratio of a stable complex eigenvalue is defined as the cosine of its angle with the negative real axis, and is a measure of how oscillatory the trajectory will be. The eigenvalues of A are determined from

$$|A - \lambda_i I| = 0, \quad i = 1, 2, \ldots, n \quad (4)$$

With these basic definitions the next section outlines the procedure for distinct eigenvalues.

3. PROCEDURE FOR DISTINCT EIGENVALUES

Assume for the time being that the system to be controlled has distinct eigenvalues. If the system matrix A varies to $A + \delta A$, then the corresponding variation in the eigenvalue is given by [Rosenbrock 1965]

$$\delta\lambda_r = \frac{\text{trace} [Q \cdot \delta A]}{\text{trace} [Q]} \quad (5)$$

where

$$Q = \prod_{\substack{i=1 \\ i \neq r}}^{n} (A - \lambda_i I) \quad (6)$$

From equation (5) it is possible to write an expression for δA in terms of $\delta\lambda_r$. If we assume that δA results due to the output feedback

of equation (2) with the gain matrix denoted by δG then

$$\delta A = B \cdot \delta G \cdot C \tag{7}$$

By substituting equation (7) in (5) and simplifying it is possible to write a linear equation for the elements of the gain matrix as

$$\sum_{k=1}^{mp} L_{rk} \left[\delta G_{ij}\right] = \delta \lambda_r \; tr \left[\lambda\right], \; r = 1, 2, \ldots . n. \tag{8}$$

where δG_{ij} is the ijth element of the matrix δG. Since there are n distinct eigenvalues, there are n independent equations. It is thus possible to write the overall equation as

$$P \cdot \delta g = V \tag{9}$$

where P is a known matrix of order n x mp, δg is the mp vector corresponding to the unknown elements of the gain matrix and V is a known constant of order n. Since the equation (5) is only valid for small perturbations, variations in the eigenvalues are applied in small steps in the desired direction to compute δg. An iterative procedure in which the gain elements were computed in small steps to achieve a desired degree of stability and damping ratio was developed and tested on several example problems.

Since the P matrix is of order nxmp, the solution of the unknown mp vector δg depends primarily on the relative magnitude of mp and n. Hence some special cases of interest are treated below:

(i) mp = n: The computation of the gain matrix is unique for a particular problem. In this case P of equation (9) is a square matrix of order n and the computation of the gain matrix is given by

$$\delta g = P^{-1} v \tag{10}$$

(ii) mp > n: In this case (mp - n) elements of the gain matrix
may assume arbitrary values and the remaining n elements are computed
as above.

(iii) mp < n: In this case we have more equations than the unknowns
and the best possible solution is the least square solution of (9) namely,

$$\delta g = (P^T P)^{-1} P^T v \tag{11}$$

However, a stable output feedback is not guaranteed for gain elements
computed this way.

A simple necessary condition proposed by Seraji (1973) gives a method
to check whether the output feedback gains will stabilize the unstable
plant or not.

4. PROCEDURE FOR MULTIPLE EIGENVALUES

The previous section illustrates the procedure for computing the
gain matrix for the case in which the eigenvalues are distinct.
Difficulty occurs for multiple eigenvalues because equation (5) is then
indeterminate. One solution is to differentiate both the numerator and
denominator terms of equation (5) with respect to the multiple eigenvalue
and use this expression in place of equation (5). The feedback gain per-
turbations are computed as before. Then using the newly computed gain
matrix, the true eigenvalues of the closed loop system matrix are
computed. It has been observed that the true eigenvalues often differ
from the expected eigenvalues; however, the sum of the changes in the
true eigenvalues is nearly equal to the sum of the perturbations
requested in the eigenvalues. When this happens, the true eigenvalues
are then distinct and the procedure illustrated in the last section may
be followed.

5. EXAMPLES

With these theoretical background it is possible to write a computer program. However, because of the complexity of handling complex matrix operations it is not possible to write a general program. Some of the examples are illustrated below:

Example 1: mp = n, distinct eigenvalues. Consider the system given by equation (1) with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \qquad (12)$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad (13)$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad (14)$$

Using the program to place the poles to the left of -1 line, the gain matrix using output feedback is given by

$$\bullet = \begin{bmatrix} 4.02 & 0 \\ -9.03 & 0 \end{bmatrix} \qquad (15)$$

and the closed loop system has the following eigenvalues:

-4; -3; -1.0053; -1.0050

Example 2: $mp > n$, distinct eigenvalues. In the previous example change the B matrix to

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{16}$$

The gain matrix obtained to place the closed loop poles to the left of -1.0 line then is

$$G = \begin{bmatrix} 4.0378 & 0.0235 \\ -9.047 & -0.0294 \\ 0.043 & 0.0 \end{bmatrix} \tag{17}$$

with the last two elements of the third row chosen arbitrarily and the closed loop poles are

-4.009118; -3.0:-1.00448; -1.001543

Example 3: $mp < n$, distinct eigenvalues. Consider the examples with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{18}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{19}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{20}$$

The open loop poles are at

$1.0: -0.5 \pm j \, 0.866025$

Using the above procedure the least square solution for the gain matrix being

$$G = \begin{bmatrix} -2.7077 & -4.0921 \end{bmatrix}$$

the closed loop poles are

$0.135627; \ -2.113864 \pm j \, 1.704338.$

Note that the least square solution does not stabilize the overall system. As illustrated before, the sign criterion proposed by Seraji fails and hence stabilization by means of output feedback matrix is not possible for this problem.

Example 4: $mp = n$, multiple eigenvalues. Consider the system

$$\dot{x}(t) = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} u(t) \tag{21}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} x(t) \tag{22}$$

The open loop poles has one unstable double pole. Using the procedure
given in section 4, the gain matrix obtained being

$$
G = \begin{bmatrix} -5.8470 & -4.35 \\ 4.4007 & 2.607 \end{bmatrix}
\tag{23}
$$

and the closed loop poles are

-3.0053; -1.9242; -1.00013; -0.996466

6. CONCLUSIONS

In summary, for a controllable, observable system an iterative
technique to determine the constant gain matrix that will stabilize
a linear constant multivariable system using output variable feedback
is presented. The method consists of adjusting iteratively the
unstable eigenvalues in a desired direction to achieve a satisfactory
degree of stability and damping ratio. From the development of the
procedure it may be seen that the method has the advantage that is
not necessary to transform variables and thereby lose physical signifi-
cance. The resultant control problem also is algebraic and simple to
compute.

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